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MIXING LAYER ON THE LEE SIDE OF AN OBSTACLE

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Two-layer miscible flow above an uneven bottom is considered. A mathematical model in the shallow-water approximation is constructed for the development of a turbulent layer between homogeneous layers of different density in a two-layer channel flow over a local obstacle. The influence of the mixing process on the formation of an initial segment of the steady-state density-stratified flow on the leeward side of the obstacle is studied.

Key words: *two-layer flow, mixing, entrainment, mixing layer, topographic effects.*

Introduction One of the interesting and poorly studied aspects of fluid flows over local obstacles is the influence of weak stratification on the global flow pattern. Among the natural phenomena due to the presence of density stratification is the development of powerful downslope flows over local rises in the atmosphere and ocean [1–3]. The main mechanism involved in the development of such flows is the generation of large vortex structures on the leeward side of the obstacle, which ensures intense vertical transfer of mass and momentum in the fluid. The separation of the boundary layer in a weakly-stratified fluid flow above an uneven bottom is also related to such phenomena [4]. In the indicated class of flows, mixing processes play a decisive role, which leads to the necessity of constructing a rather simple mathematical model for these phenomena.

In the present paper, we analyze a mathematical model for two-layer shallow water above an uneven bottom taking into account the turbulent mixing of the layers [5]. This model extends the possibilities of applying classical shallow-water theory to a wide class of turbulent stratified flows. The model constructed is simple enough to be used in analytical studies of the evolution of the main types of flows, such as mixing layers, buoyant jets, and density or gravity flows. The problem of the formation of a mixing layer in a two-layer miscible flow over a slope is considered as an example of application of the model.

1. Mathematical Model. The two-layer shallow-water equations taking into account turbulent mixing between the layers [5] for flows above an uneven bottom can be written as

$$h_t + (hu)_x = -\chi^-, \qquad (\zeta)_t + (\zeta w)_x = -\chi^+, \qquad \eta_t + (\eta v)_x = \bar{\chi},$$

$$u_t + (u^2/2 + bh + \bar{b}\eta + p)_x = -bz_x, \quad w_t + (w^2/2 + p)_x = 0, \quad (bh + \bar{b}\eta)_t + (bhu + \bar{b}\eta v)_x = 0,$$

$$(hu + \eta v + \zeta w)_t + \Big(hu^2 + \eta v^2 + \zeta w^2 + bh^2/2 + \bar{b}\eta h + \bar{b}\eta^2/2 \tag{1}$$

$$+ (h + \eta + \zeta)p\Big)_x = -(p + bh + \bar{b}\eta)z_x,$$

$$(hu^2 + \eta(v^2 + q^2) + \zeta w^2 + bh^2 + 2\bar{b}\eta h + \bar{b}\eta^2)_t + \Big(hu^3 + \eta v(v^2 + q^2) + \zeta w^3$$

$$+ 2p(hu + \eta v + \zeta w) + 2\bar{b}\eta hu + 2\bar{b}(h + \eta)\eta v + 2bh^2 u)_x = -2(bhu + \bar{b}\eta v)z_x - \varepsilon.$$

Here h, ζ, and η are the thicknesses of the lower and upper layers and the interlayer, respectively, $z(x)$ is the shape of the bottom, u, w, and v are the corresponding velocities in the layers, $b = (\rho^- - \rho^+)g/\rho^+$ is the specified buoyancy

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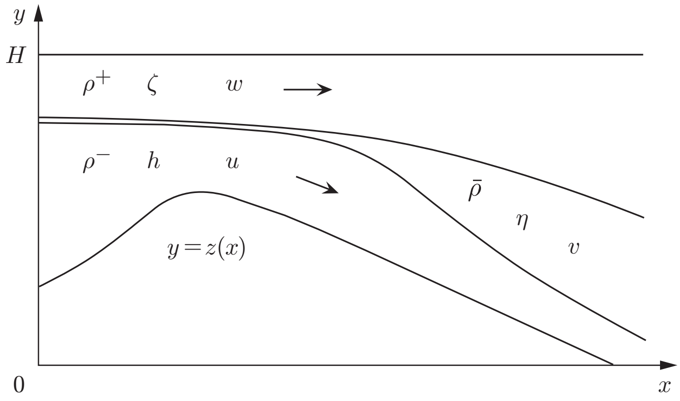

Fig. 1. Diagram of a two-layer flow over an obstacle.

of the lower layer, $\bar{b} = (\bar{\rho} - \rho^+)g/\rho^+$ is the buoyancy in the interlayer, ρ^-, ρ^+, and $\bar{\rho}$ are the corresponding densities of the layers, g is the acceleration due to gravity, p is the specific pressure on the upper boundary of the flow, q^2 is the specific kinetic energy of the "large vortices" determining the entrainment of the fluid from the homogeneous layers into the interlayer (Fig. 1). The rates of entrainment χ^+, χ^-, and $\bar{\chi} = \chi^+ + \chi^-$ are considered known functions of the sought-for variables. In the Boussinesq approximation, the total depth $H = h + \eta + \zeta + z \equiv \text{const}$. Because the fluid is incompressible, it follows that the total consumption $hu + \eta v + \zeta w$ is a function of only time and is specified by virtue of the boundary conditions. Energy dissipation in the turbulent interlayer is taken into account by specifying the function ε in (1). The choice of the functions $\chi^\pm$ and ε for different types of flow on an underwater slope will be performed below.

A more detailed insight into the structure of system (1) can be obtained from the differential form of this system:

$$h_t + uh_x + hu_x = -\chi^-, \qquad \zeta_t + w\zeta_x + \zeta w_x = -\chi^+, \qquad \eta_t + v\eta_x + \eta v_x = \chi^+ + \chi^-,$$

$$u_t + uu_x + bh_x + \bar{b}\eta_x + \eta\bar{b}_x + p_x = -bz_x, \qquad w_t + ww_x + p_x = 0, \tag{2}$$

$$v_t + vv_x + \bar{b}h_x + \bar{b}\eta_x + \eta\bar{b}_x/2 + p_x = -\bar{b}z_x + (\chi^-(u - v) + \chi^+(w - v))/\eta;$$

$$\bar{b}_t + v\bar{b}_x = (\chi^-(b - \bar{b}) - \chi^+\bar{b})/\eta,$$

$$q_t + vq_x = (\chi^-((u - v)^2 - q^2 - (b - \bar{b})\eta) + \chi^+((w - v)^2 - q^2 - \bar{b}\eta) - \varepsilon)/(2\eta q). \tag{3}$$

Subsystem (2) is an inhomogeneous system of three-layer shallow-water equations in the Boussinesq approximation, and the left side of Eqs. (3) is the derivative along the particle trajectory in the interlayer. Therefore, characteristics (1) coincide with the characteristics of the three-layer shallow-water equations and have an additional multiple contact characteristic [5] $dx/dt = v$. The main advantage of the approach formulated above is that within the framework of a three-layer flow pattern, the use of the complete conservation laws of mass, momentum, and energy allows one not only to close the system and uniquely find relations on discontinuities but also to obtain particular expressions for the right sides of Eqs. (2) and (3), which describe particle entrainment into the turbulent interlayer due to the development of shear instability of the flow.

To close model (1), it is necessary to specify the functions $\chi^\pm$ and ε. The representation of the dissipation law as

$$\varepsilon = \mu q^3, \qquad \mu \equiv \text{const} \tag{4}$$

follows from the choice of the interlayer thickness η as the characteristic turbulence scale. In choosing the law of entrainment into the interlayer, the simple hypothesis on the proportionality of the vertical transfer velocity to the characteristic velocity of the vortex structures generated by a velocity shear in the mixing layer leads to the following dependence:

$$\chi^\pm = \sigma q, \qquad \sigma \equiv \text{const}. \tag{5}$$

Relation (5) with $\sigma = 0.15$ was experimentally supported in [6] for homogeneous free shear flows. In stratified flows, the entrainment law (5) adequately describes the nonlinear stage of development of Kelvin–Helmholtz instability in mixing layers, gravity flows in horizontal and inclined channels, and the other basic types of flows determined by turbulent mixing processes [5]. If, as a result of evolution, the turbulent interlayer approaches one of the external boundaries of the flow (the bottom, rigid boundary, or free surface), the entrainment ceases (χ^+ or χ^- vanishes) and the mixing layer becomes a submerged jet. With an appropriate choice of the entrainment law, the mathematical model (1) describes this flow transformation.

Next, we consider the development of a steady-state mixing layer in a supercritical two-layer flow over an inclined plane. Since we study the initial stage of the process before the moment the mixing layer reaches the bottom or the free surface, the flow is described using Eqs. (1), (4), and (5). Equations (1) is also applicable for describing:

— surface jets $\chi^+ = 0$ and $\chi^- = \sigma q$;

— near-bottom layers $\chi^- = 0$ and $\chi^+ = \sigma q$.

The indicated models, providing an adequate characterization of important hydrodynamic processes such as boundary-layer separation and the development of a powerful slope stream in weakly stratified flows over an underwater obstacle require an additional analysis and are not considered in the present paper.

2. Steady-State Two-Layer Flows. We consider the evolution of a mixing layer in a channel of variable depth (see Fig. 1). Intense mixing between the layers arises on the leeward side of the obstacle in the case where a subcritical two-layer flow ahead of the obstacle is transformed into a supercritical flow behind it. The notion of flow supercriticality and subcriticality is related to the velocity of long internal waves in a two-layer flow. The experiments of [7] show that in a two-layer miscible flow on the leeward side of an obstacle, the upper layer is decelerated and the lower layer is accelerated. The development of shear instability on the interface between the layers leads to the formation of a mixing layer, in which the development of large vortex structures in the interlayer is maintained by the rearrangement of the velocity profile. Because this mechanism of development of a turbulent interlayer in two-layer flows underlies the model considered, the evolution of the mixing layer above the slope can be described by the following equations of steady-state two-layer miscible flows above an uneven bottom:

$$
\begin{gathered}
uh_x + hu_x = -\chi^-, \qquad w\zeta_x + \zeta w_x = -\chi^+, \qquad v\eta_x + \eta v_x = \chi^- + \chi^+, \\
uu_x + bh_x + \bar{b}\eta_x + \eta\bar{b}_x + p_x = -bz_x, \qquad ww_x + p_x = 0, \\
vv_x + \bar{b}h_x + \bar{b}\eta_x + \eta\bar{b}_x/2 + p_x = -\bar{b}z_x + (\chi^-(u-v) + \chi^+(w-v))/\eta, \quad \eta v\bar{b}_x = \chi^-(b-\bar{b}) - \chi^+\bar{b}, \\
2\eta v q q_x = \chi^-\Big((u-v)^2 - q^2 - (b-\bar{b})\eta\Big) + \chi^+\Big((w-v)^2 - q^2 - \bar{b}\eta\Big) - \varepsilon.
\end{gathered}
\tag{6}
$$

For $\eta > 0$, system (6) can be solved for the derivatives if the determinant

$$
\Delta = (u^2/h + w^2/\zeta - b)(v^2/\eta + w^2/\zeta - \bar{b}) - (w^2/\zeta - \bar{b})^2
$$

does not vanish. In this case,

$$
\begin{gathered}
h_x = (a_1(v^2/\eta + w^2/\zeta - \bar{b})/h - a_2(w^2/\zeta - \bar{b})/\eta)/\Delta\ , \qquad u_x = -\chi^-/h - uh_x/h, \\
\eta_x = \Big(a_1 - (u^2 - bh + hw^2/\zeta)h_x\Big)/(hw^2/\zeta - \bar{b}h), \qquad v_x = (\chi^+ + \chi^- - v\eta_x)/\eta, \\
w_x = \Big(-\chi^+ + w(h_x + \eta_x + z_x)\Big)/\zeta, \qquad \zeta_x = -(\chi^+ + \zeta w_x)/w,
\end{gathered}
\tag{7}
$$

where $a_1 = h(b - w^2/\zeta)z_x + \chi^+ hw/\zeta - \chi^- u + (\chi^-(b-\bar{b}) - \chi^+\bar{b})h/v$, $a_2 = \eta(\bar{b} - w^2/\zeta)z_x + \chi^+(2v - w) + \chi^-(2v - u) + \chi^+\eta w/\zeta + (\chi^-(b-\bar{b}) - \chi^+\bar{b})\eta/(2v)$. The derivatives $\bar{b}_x$ and q_x are determined in (6).

System (6) describes a wide class of steady-state flows above an uneven bottom in which mixing on the interface between two homogeneous layers is a determining factor for flow formation. The supercritical slope flows considered below belong to such flows.

3. Mixing Layer above a Slope. We consider the generation of a turbulent layer in a supercritical two-layer flow in a channel of variable depth. The formation of a mixing layer due to the development Kelvin–Helmholtz instability is an important element of many stratified flows over obstacles in both the atmosphere and the upper layer of the ocean. If an obstacle controls the upstream flow, a subcritical flow occurs ahead of the obstacle, in

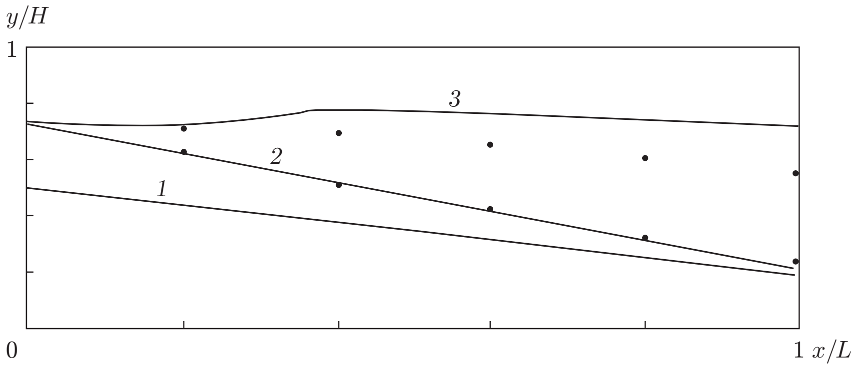

Fig. 2. Development of a mixing layer on an inclined plane: curve 1 refers the to channel bottom and curves 2 and 3 refer to the lower and upper boundary of the mixing layer, respectively [calculation for model (1)]; the points show the corresponding boundaries of the mixing layers found from the linear part of the velocity profile in the mixing layer [7].

which mixing between the layers is insignificant and a two-layer flow pattern is quite adequate. On the leeward side of the obstacle, a supercritical flow regime develops, and in the neighborhood of the interface between the layers, a region of intense mixing is formed. The experiments of [7] revealed that the development of instability of a two-layer flow corresponds to the evolution of a mixing layer in free shear flows [6]. In the initial stage, the thickness of the mixing layer increases linearly with increase in the distance from the top of the obstacle, as in a channel of constant depth; then, the rate of entrainment of the fluid from the homogeneous layers into the interlayer decrease suddenly (Fig. 2). This flow segment is formed as a result of acceleration of the lower fluid layer along the slope. Next, the mixing layer reaches the bottom, and the flow is transformed into a near-bottom turbulent jet. We note that the initial segment of the mixing-layer development determines the main characteristics of the near-bottom jet, such as the total buoyancy flux and the flow rate in the core of the density-stratified flow [5]. We use model (1), (4), (5) to describe the evolution of the initial segment of the mixing layer in the following formulation.

On the left boundary of the channel ($x = 0$), let a supercritical two-layer flow be specified:

$$\eta_0 = 0, \qquad h = h_0 > 0, \qquad \zeta = \zeta_0 > 0, \qquad u = u_0 > 0, \qquad w = w_0 \tag{8}$$

and $\Delta_0 = u_0^2/h_0 + w_0^2/\zeta_0 - b > 0$. At a small distance from the left boundary, solution (4)–(6) is determined by the following asymptotic form of the mixing layer, which ensures boundedness of the derivatives for $x = 0$ [5]:

$$\bar{b}_0 = b/2, \qquad v_0 = (u_0 + w_0)/2, \qquad q_0 = |u_0 - w_0|/\sqrt{2(2 + \mu/\sigma)}. \tag{9}$$

By virtue of the symmetry of the entrainment law (4) and relations (9), the average buoyancy in the interlayer is constant: $\bar{b} \equiv \bar{b}_0 = b/2$.

Next, in the mixing layer, the entrainment from the homogeneous layers into the interlayer begins on the left boundary of the channel $x = 0$ at a finite rate $\chi^{\pm} = \sigma q_0$. Therefore, at $x > 0$, the thickness of the interlayer η is positive and solution (7) can be found in a certain domain $x > 0$. In particular, for the initial data (8), which correspond to the laboratory experiment of [7], the supercritical flow ($\Delta > 0$) can be constructed up to the moment the mixing layer becomes a near-bottom jet ($h = 0$).

Figure 2 shows the solution of problem (7)–(9) that describes the development of a mixing layer above an inclined plane in a two-layer miscible flow implemented in [7]. The point $x = 0$ corresponds to the beginning of development of Kelvin–Helmholtz instability in supercritical flow and is at a distance of about 10 cm from the top of the local obstacle in the experiment. In [7], it is shown that on the segment between the top and the point at which the mixing layer begins to form, the critical flow becomes supercritical without substantial mixing between the layers.

The solution of problem (7)–(9) ($\sigma = 0.15$ and $\mu = \sigma$) is constructed for the following initial and boundary conditions: $H = 30$ cm, $L = 50$ cm, $z_0 = 15$ cm, $h_0 = 7$ cm, $u_0 = 5$ cm/sec, $w_0 = 0$, $b = 1.4$ cm/sec^2, and $\varphi = 10.8°$ (φ is the slope of the plane to the horizon). In the solution, it is possible to distinguish the initial segment of the mixing-layer development, in which buoyancy effects have little influence on the mixing process, and the main segment of the turbulent-layer development due to acceleration of the lower homogeneous layer. The structure of the solution qualitatively corresponds to the structure of the flow studied in [7]. The points in Fig. 2 show the mixing-layer boundaries found from the experimental data on the velocity distribution in the mixing layer given in [7, Fig. 9]. The thickness of the turbulent layer was determined form the linear segment of the velocity distribution in the interlayer. We note that model (1) specifies the external boundaries of the turbulent-mixing region occupied by "large vortices." The effective thickness of the mixing region in which the velocity and density profiles are nearly linear is much smaller than the total thickness found from (1).

A similar situation arises in the simulation of a mixing layer in a homogeneous fluid using (1) [5]. The exact solution (1), describing the development of a self-similar mixing layer, gives a rate of increase of the mixing region $d\eta/dx = 2\sigma = 0.3$, and the experimental estimate of the rate of increase in the effective thickness of the mixing layer is $d\eta/dx = 0.18$ As shown in [5, Fig. 7.4], the theoretical value for the law of increase in the effective thickness $d\eta/dx = 0.18$ in the mixing layer can be obtained using model (1) as follows: the dependences $\eta = \eta(x)$ and $q = q(x)$ obtained with the use of (1) are then employed as the linear scale and the turbulence scale in the semiempirical model of the next level to find the distributions of the longitudinal velocity component and Reynolds stresses in the mixing layer. Then, the effective boundaries of the turbulent flow are obtained from the velocity profile. This approach can also be used to find the velocity profile in turbulent stratified flows behind obstacles but this is beyond the scope of the present paper.

In Eqs. (1), the effect of the near-bottom boundary layer due to flow friction on the channel bottom was ignored. This assumption is valid until the thickness of the boundary layer is smaller than the thickness of the lower homogeneous layer. When the low boundary of the mixing layer reaches the bottom ($h \to 0$), the gravity flow becomes a turbulent jet with a rather complex internal structure, the entrainment into the interlayer is no longer symmetric, and hypothesis (4) becomes inadequate. A simple mathematical model for an unsteady density-stratified flow over an inclined plane including the evolution of a turbulent layer of variable density and the presence of a near-bottom flow core of constant density is presented in [5]. We note that within the framework of this model, the necessary boundary conditions for the calculation of both the main steady-state segment of the density-stratified flow and the unsteady head of the flow can be found from the solution of the above problem of the formation of the initial segment of a mixing layer above a slope.

In conclusion, we note that with an appropriate choice of the entrainment laws $\chi^{\pm}$, model (1) describes different stages of formation of an intense turbulent mixing region in stratified flows over obstacles. This problem has numerous applications in oceanology and meteorology, for example, in the analysis of underwater falling streams, powerful downslope flows, etc. Model (1) is a rather simple instrument for studying natural flows. It allows a comparison of analytic solutions with real flow patterns and is an intermediate model between classical shallow-water theory and the direct numerical modeling of turbulent flows.

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